Europ. J. Combinatorics (1980) **1**, 195–199

Hadwiger's Conjecture is True for Almost Every Graph

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The contraction clique number ccl(G) of a graph G is the maximal r for which G has a subcontraction to the complete graph K^r. We prove that for $d > 2$, almost every graph of order n satisfies $n((\log_2 n)^{\frac{1}{2}}+4)^{-1} \leqslant \mathrm{ccl}(G) \leqslant n(\log_2 n - d \log_2 \log_2 n)^{-\frac{1}{2}}$. This inequality implies the statement in the title.

1. Introduction

One of the deepest unsolved problems in graph theory is the following conjecture due to Hadwiger [7]: $\chi(G) = s$ implies $G > K^s$. In other words, every s-chromatic graph G has a subcontraction to K^s, the complete graph of order s. In the case $s = 5$, this is equivalent to the four-colour theorem. (For an account of the various results related to Hadwiger's conjecture the reader is referred to [1, Chapter VII]; the terminology and notation not defined here can also be found in [1].)

The statement in the title would sound rather hollow but for certain recent developments. Hajós conjectured that every s-chromatic graph contains a TK^s, a topological complete subgraph of order s, that is a subdivision of K^s. This is clearly stronger than Hadwiger's conjecture, for a TK^s itself has a contraction to K^s, but a graph subcontractible to K^s need not contain a TK^s. The Hajós conjecture was disproved recently by Catlin [5], who exhibited counter-examples for $\chi(G) \geqslant 7$. Shortly after Catlin's result Erdös and Fajtlowicz [6] showed that almost every graph is a counter-example to the Hajós conjecture. More precisely, define the *topological clique number* of a graph G as

$$\mathrm{tcl}(G) = \max\{r \colon G \supset TK^r\}.$$

Erdös and Fajtlowicz showed that for almost every graph G of order n,

$$\mathrm{tcl}(G) \leqslant cn^{\frac{1}{2}} \tag{1}$$

for some absolute constant c. Since for every $\varepsilon > 0$ almost every graph satisfies

$$\chi(G) \geqslant (\tfrac{1}{2} - \varepsilon)n/\log_2 n,$$

we have that

$$\mathrm{tcl}(G) < \chi(G)$$

for almost every graph (for sharp results on $\chi(G)$ see [4]).

Inequality (1) was extended by Bollobás and Catlin [3], who proved that for every $\varepsilon > 0$ almost every graph satisfies

$$(2-\varepsilon)n^{\frac{1}{2}} \leqslant \mathrm{tcl}(G) \leqslant (2+\varepsilon)n^{\frac{1}{2}} \tag{2}$$

and so

$$(\tfrac{1}{4} - \varepsilon)n^{\frac{1}{2}}/\log_2 n \leqslant \chi(G)/\mathrm{tcl}(G).$$

In view of this it is imperative to attack Hadwiger's conjecture by random graphs, that is to examine whether or not Hadwiger's conjecture holds for almost every graph. This is

* This work was supported by the National Science Foundation under Grant No. MCS–7903215.

0195–6698/80/030195+05 $01.00/0 © 1980 Academic Press Inc. (London) Limited

exactly the task we shall accomplish in this note. More precisely, we shall prove an analogue of (2) for the *contraction clique number* $\mathrm{ccl}(G)$ of a graph G, defined as

$$\mathrm{ccl}(G)=\max\{r:\ G>K^r\}.$$

2. Random Graphs

Let $0<p<1$ be fixed, and let V be a set of n distinguishable vertices. Denote by $\mathscr{G}(n, P(\text{edge})=p)$ the discrete probability space consisting of all graphs with vertex set V, in which the probability of a graph of size m is

$$p^m(1-p)^{\binom{n}{2}-m}.$$

In other words, the edges of a graph $G\in\mathscr{G}(n, P(\text{edge})=p)$ are chosen independently and with probability p. (See [2, Chapter VII] for results concerning this model.)

Given a property $\mathscr{P}$ of graphs we define the *probability of* $\mathscr{P}$ as

$$P(\mathscr{P})=P(\{G\in\mathscr{G}(n, P(\text{edge})=p):\ \mathscr{P}\text{ holds for }G\}).$$

If $P(\mathscr{P})\to 1$ as $n\to\infty$ then the property $\mathscr{P}$ is said to hold for *almost every* graph.

In order to make the calculations below a little more pleasant, we shall take $p=\frac{1}{2}$. The case $p=\frac{1}{2}$ is in some sense the most natural, since this is the model one considers implicitly when one counts the proportion of all graphs having a given property. Indeed, in the model $\mathscr{G}=\mathscr{G}(n, P(\text{edge})=\frac{1}{2})$ every graph has the same probability, so the probability of a set $\mathscr{H}\subset\mathscr{G}$ is exactly $|\mathscr{H}|/|\mathscr{G}|$. Thus a property $\mathscr{P}$ holds for almost every graph in $\mathscr{G}(n, P(\text{edge})=\frac{1}{2})$ iff the number of graphs having $\mathscr{P}$ is asymptotically equal to the number of all graphs (with vertex set V).

3. The Contraction Clique Number

Given a graph G and non-empty disjoint subsets $V_1, V_2, \ldots, V_s$ of $V=V(G)$, denote by $G/\{V_1,\ldots,V_s\}$ the graph with vertex set $\{V_1, V_2, \ldots, V_s\}$ in which V_i is joined to V_j iff G contains a V_i-V_j edge. Put

$$\mathrm{ccl}'(G)=\max\{r:\ G/\{V_1,\ldots,V_r\}\cong K^r \text{ for some } V_1,\ldots,V_r\}.$$

Since the contraction clique number is defined similarly, except with the added restriction on the V_i that each $G[V_i]$ is connected,

$$\mathrm{ccl}(G)\leqslant\mathrm{ccl}'(G).$$

We shall give a lower bound for $\mathrm{ccl}(G)$ and an upper bound for $\mathrm{ccl}'(G)$ holding for almost every graph. As customary, $\log_b x$ denotes the logarithm to base b.

Theorem. *Let $d>2$. Then almost every graph $G\in\mathscr{G}(n, P(\text{edge})=\frac{1}{2})$ satisfies*

$$n((\log_2 n)^{\frac{1}{2}}+4)^{-1}\leqslant\mathrm{ccl}(G)\leqslant\mathrm{ccl}'(G)$$
$$\leqslant n(\log_2 n-d\log_2\log_2 n)^{-\frac{1}{2}}\leqslant n((\log_2 n)^{\frac{1}{2}}-1)^{-1}.$$

Proof. (a) We start with a proof of the upper bound on $\mathrm{ccl}'(G)$. Put $s=\lfloor n(\log_2 n-d\log_2\log_2 n)^{-\frac{1}{2}}\rfloor$. A partition $\{V_1, V_2,\ldots, V_s\}$ of the vertex set V is said to be *permissible for a graph G* if G contains a V_i-V_j edge for every pair (i,j), $1\leqslant i<j\leqslant s$. Thus $\mathrm{ccl}'(G)\geqslant s$ iff the graph G has a permissible partition. We have to prove that the probability that a graph has a permissible partition tends to 0 as $n\to\infty$.

To start with, note rather crudely that there are at most

$$\frac{n!}{s!}\binom{n}{s-1}<n^n \tag{3}$$

partitions of V into s non-empty sets. The number on the left-hand side of (3) is the number of partitions of V into s non-empty *ordered* sets.

Consider now a fixed partition $\mathcal{P}=\{V_1, V_2, \ldots, V_s\}$ into non-empty sets. What is the probability that this partition $\mathcal{P}$ is permissible? Let $n_1, n_2, \ldots, n_s$ be the number of vertices in the classes. Then the probability that a graph contains no V_i-V_j edge is $2^{-n_in_j}$. Hence

$$P(\mathcal{P} \text{ is permissible}) = \Pi(1-2^{-n_in_j}) \leqslant e^{-\Sigma 2^{-n_in_j}}, \tag{4}$$

where both the product and the sum are taken over all pairs (i,j) with $1 \leqslant i<j\leqslant s$. We have the following string of elementary inequalities.

$$\Sigma 2^{-n_in_j}\binom{s}{2}^{-1} \geqslant (\Pi 2^{-n_in_j})^{\binom{s}{2}^{-1}} = 2^{-(\Sigma n_in_j)\binom{s}{2}^{-1}} \geqslant 2^{-n^2/s^2}. \tag{5}$$

The reader may note that Σn_in_j is exactly the number of edges in the complete s-partite graph with vertex classes $V_1, V_2, \ldots, V_s$. The Turán graph $T_s(n)$ is the unique s-partite graph with maximal number of edges, and

$$e(T_s(n)) = \left(\frac{s-1}{2s}+o(1)\right)n^2 \qquad \text{(see [2, p. 71])}.$$

From (4) and (5) we have

$$P(\mathcal{P} \text{ is permissible}) \leqslant e^{-\binom{s}{2}2^{-n^2/s^2}}, \tag{6}$$

and (3) and (6) imply

$$P(G \text{ has a permissible partition} = P(\mathrm{ccl}'(G)\geqslant s) \leqslant n^n e^{-\binom{s}{2}2^{-n^2/s^2}}$$
$$= P_s. \tag{7}$$

Clearly

$$\log P_s = n\log n - \binom{s}{2}2^{-n^2/s^2} \leqslant n\left\{\log n - \frac{1}{3\log_2 n}2^{d\log_2\log_2 n}\right\} \leqslant -\tfrac{1}{4}n(\log_2 n)^{d-2} \to -\infty.$$

Hence $P_s \to 0$, proving the required upper bound on $\mathrm{ccl}'(G)$.

(b) We turn to the proof of the lower bound on $\mathrm{ccl}(G)$. Put $k=\lceil(\log n)^{\frac{1}{2}}+\frac{1}{2}\rceil$, $s=\lceil n/(k^5/2)\rceil$ and $t=\lfloor n/(k+2)\rfloor$. We shall prove in two steps that $G>K^s$ for almost every graph G.

Step 1. Fix a set T of t vertices and put $W=V-T$. Then *almost every graph G contains t vertex disjoint stars of order $k+1$ whose centres are the t vertices in T.*

Indeed, by a slight extension of Hall's theorem (see [2, p. 56]) if G does not contain such stars then there is a set $A\subset T$ for which the vertices in A have less than $k|A|$ neighbours in W. Given a set A with $a=|A|$ elements, the probability that a vertex in W is joined to no vertex in A is 2^{-a}. Hence the probability that the vertices in A have less than ka neighbours in W is at most

$$\sum_{u<ka}\binom{n-t}{u}2^{-a(n-t-u)} < n^{ka}2^{-a(n-t-ka)}$$
$$\leqslant n^{ka}2^{-at} < 2^{-at/2}.$$

Consequently the probability that G fails to contain the desired t stars is at most

$$\sum_{a \leq t} \binom{t}{a} 2^{-at/2} \leq \sum_{a \leq t} (t2^{-t/2})^a \leq 2t2^{-t/2},$$

and this tends to 0.

Step 2. Let $V_1, V_2, \ldots, V_t$ be the vertex sets of the stars constructed in Step 1 in almost every graph. Then for *almost every graph* G *there are* $V_{n_1}, V_{n_2}, \ldots, V_{n_s}$ *such that* $G/\{V_{n_1}, V_{n_2}, \ldots, V_{n_s}\} \cong K^s$. The assertions in these two steps clearly imply the first inequality of our theorem.

Note that the sets $V_1, V_2, \ldots, V_t$ depend only on the $T-W$ edges of the graph. Thus the edges joining the vertices of W are chosen independently with probability $\frac{1}{2}$. Put $W_i = V_i - T$. We say that (W_i, W_j), $i \neq j$, is *good* if there is a $W_i - W_j$ edge. Since $W_i \subset W$ and $|W_i| = k$, clearly

$$P(\text{the pair } (W_i, W_j) \text{ is bad}) = 2^{-k^2}$$

and so the expected number of bad pairs is

$$\binom{t}{2} 2^{-k^2} < \frac{n^2}{\log_2 n} 2^{-\log_2 n - (\log_2 n)^{\frac{1}{2}}} = \frac{n}{\log_2 n} 2^{-(\log_2 n)^{\frac{1}{2}}}.$$

At this stage we have several options. We may appeal either to the classical De Moivre–Laplace theorem (see [2; p. 134]) or to the even simpler Chebyshev inequality (see [2, p. 134]) or to the trivial inequality $\mathrm{P}(|\mathrm{X}| \geq |\mathrm{c}|) \leq \mathrm{E}(|\mathrm{X}|)/|\mathrm{c}|$ to deduce that almost every graph has few bad pairs. For example, the last inequality implies that the probability that a graph has more than

$$\frac{n}{\log_2 n} 2^{-\frac{1}{2}(\log_2 n)^{\frac{1}{2}}}$$

bad pairs is at most $2^{-\frac{1}{2}(\log_2 n)^{\frac{1}{2}}}$. In particular, since

$$t - \frac{n}{\log_2 n} 2^{-\frac{1}{2}(\log_2 n)^{\frac{1}{2}}} > s,$$

for almost every graph we can find sets $W_{n_1}, W_{n_2}, \ldots, W_{n_s}$ such that every pair (W_{n_i}, W_{n_j}) is good. Then we have $G/\{V_{n_1}, \ldots, V_{n_s}\} \cong K^s$ and since each $G[V_i]$ is connected, $\mathrm{ccl}(G) \geq s$, as claimed.

The proof of our theorem is complete.

With a little more effort the lower bound can be improved to $n((\log_2 n)^{\frac{1}{2}} + 1)^{-1}$. Furthermore, the calculations can easily be carried over to the general case. If $0 < p < 1$ is fixed then almost every graph in $\mathcal{G}(n, P(\text{edge}) = p)$ satisfies the inequality in the Theorem, with $\log_2 n$ replaced by $\log_b n$, where $b = 1/q$.

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Received 23 October 1979 and in revised form 6 December 1979

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